# Localization of electromagnetic waves in a two-dimensional random medium

Zhen Ye,* Sheng Li, and Xin Sun

*Research Center for Theoretical Physics, Fudan University, Shangai, China*
*and National Laboratory of Infrared Physics, Shanghai, China*

(Received 19 February 2002; published 7 October 2002)

Motivated by previous investigations on the radiative effects of the electric dipoles embedded in structured cavities, localization of electromagnetic waves in two dimensions is studied *ab initio* for a system consisting of many randomly distributed two-dimensional dipoles. A set of self-consistent equations, incorporating all orders of multiple scattering of the electromagnetic waves, is derived from first principles and then solved numerically for the total electromagnetic field. The results show that spatially localized electromagnetic waves are possible in such a simple but realistic disordered system. When localization occurs, a coherent behavior appears and is revealed as a unique property differentiating localization from either the residual absorption or the attenuation effects.

When propagating through a medium consisting of many scatterers, waves will be scattered by each scatterer. The scattered waves will be again scattered by other scatterers. Such a process will be repeated to establish an infinite recursive pattern of multiple scattering. As a result, the wave propagation may be significantly altered [1]. It is now well known that the multiple scattering of waves is responsible for many fascinating phenomena, ranging from phenomena of macroscopic scales such as twinkling lights in the evening sky, modulation of ambient noise in the oceans, and electromagnetic scintillation of turbulence in the atmosphere, to phenomena of microscopic or mesoscopic scales including random lasers [2], and electronic resistivity in disordered solids. It has also been proposed that under certain conditions, the multiple scattering can lead to the unusual phenomenon of wave localization, a concept originally introduced to describe disorder induced metal-insulator transitions in electronic systems [3].

Over the past two decades, localization of classical waves has been under intensive investigations, leading to a very large body of literature (e.g., Refs. [4–20]). Such a localization phenomenon has been characterized by two levels. One is the weak localization associated with the enhanced backscattering. That is, waves which propagate in the two opposite directions along a loop will obtain the same phase and interfere constructively at the emission site, thus enhancing the backscattering. The second is the strong localization, without confusion often just termed as localization, in which a significant inhibition of transmission appears, and the energy is mostly confined spatially in the vicinity of the emission site.

While the weak localization, regarded as a precursor to the strong localization, has been well studied both theoretically (e.g., the monograph in Ref. [17]) and experimentally (e.g., Ref. [18]), observation of strong localization of classical waves for higher than one dimension remains a subject of

debate [19–21], primarily because a suitable system is hard to find, and the observation is often obscured by such effects as the residual absorption [21] and scattering attenuation.

In this paper, we wish to present a simple but realistic system to study the phenomenon of strong localization in two dimensions. The system we consider stems from the previous research on enhancement and inhibition of electromagnetic radiation in structured media. For example, Kuhn [22] used the model to describe the effect of a metallic mirror on the radiation from a nearby excited molecule, whereas Chance *et al.* [23], used to explain the experimental data of Drexhage. Later, Erdogan *et al.* [24], and Wang, Ye [25] employed the model to analyze the effects of the cylindrically periodic structure on the radiation of an enclosed two-dimensional (2D) dipole. Inspired by the work of Erdogan *et al.* [24], we construct a system consisting of many 2D dipoles. The propagation of electromagnetic (EM) waves in such a system is formulated rigorously in terms of a set of coupled equations, and then is solved numerically. We show that a strong localization of EM waves is possible in this system. In line with the work on acoustic localization [26], a phase diagram method is used to describe localization of EM waves. Since the system considered here results from the practical application of light emission in structures, an experimental verification may be readily realized.

Following Erdogan *et al.* [24], we consider 2D dipoles as an ensemble of harmonically bound charge elements. In this way, each 2D dipole is regarded as a single dipole line, characterized by the charge and dipole moment per unit length. Assume that $N$ parallel dipole lines, aligned along the $z$ axis, are embedded in a uniform dielectric medium and *randomly* located at $\vec{r}_i$ ($i = 1, 2, \ldots, N$). The averaged distance between dipoles is $d$. A stimulating dipole line source is located at $\vec{r}_s$, transmitting a continuous wave of angular frequency $\omega$. By the geometrical symmetry of the system, we only need to consider the $z$ component of the electrical waves.

Upon stimulation, each dipole will radiate EM waves. The radiated waves will then repeatedly interact with the dipoles, forming a process of multiple scattering. The equation of motion for the $i$th dipole is

———
*On leave from Department of Physics, National Central University, Taiwan. Electronic address: zhen@phy.ncu.edu.tw

$$\frac{d^2}{dt^2}p_i + \omega_{0,i}^2 p_i = \frac{q_i^2}{m_i}E_z(\vec{r}_i) - b_{0,i}\frac{d}{dt}p_i \quad \text{for } i=1,2,\ldots,N,$$
(1)

where $\omega_{0,i}$ is the resonance (natural) frequency, $p_i$, $q_i$, and $m_i$ are the dipole moment, charge, and effective mass per unit length of the $i$th dipole, respectively. $E_z(\vec{r}_i)$ is the total electrical field acting on dipole $p_i$, which includes the radiated field from other dipoles and also the field directly from the source. The factor $b_{0,i}$ denotes the damping due to energy loss and radiation, and can be determined by energy conservation. Without energy loss (to such as heat), $b_{0,i}$ can be determined from the balance between the radiative and vibrational energies, and is given as [24]

$$b_{0,i} = \frac{q_i^2 \omega_{0,i}}{4\epsilon m_i c^2},$$
(2)

with $\epsilon$ being the constant permittivity and $c$ is the speed of light in the medium separately.

Equation (1) is virtually the same as Eq. (1) in Ref. [24]. The only difference is that in Ref. [24], $E_z$ is the reflected field at the dipole due to the presence of reflecting surrounding structures, while in the present case the field is from the stimulating source and the radiation from all other dipoles.

The transmitted electrical field from the continuous line source is determined by the Maxwell equations [24]

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)G_0(\vec{r}-\vec{r}_s) = -4\mu_0\omega^2 p_0\pi\delta(\vec{r}-\vec{r}_s)e^{-i\omega t},$$
(3)

where $\omega$ is the radiation frequency, and $p_0$ is the source strength and is set to be unit. The solution of Eq. (3) is clearly

$$G_0(\vec{r}-\vec{r}_s) = (\mu_0\omega^2 p_0)i\pi H_0^{(1)}(k|\vec{r}-\vec{r}_s|)e^{-i\omega t},$$
(4)

with $k=\omega/c$, and $H_0^{(1)}$ being the zeroth order Hankel function of the first kind.

Similarly, the radiated field from the $i$th dipole is given by

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)G_i(\vec{r}-\vec{r}_i) = \mu_0\frac{d^2}{dt^2}p_i\delta^{(2)}(\vec{r}-\vec{r}_i).$$
(5)

The field arriving at the $i$th dipole is composed of the direct field from the source and the radiation from all other dipoles, and thus is given as

$$E_z(\vec{r}_i) = G_0(\vec{r}_i-\vec{r}_s) + \sum_{j=1,j\neq i}^N G_j(\vec{r}_i-\vec{r}_j).$$
(6)

Substituting Eqs. (4), (5), and (6) into Eq. (1), and writing $p_i = p_i e^{-i\omega t}$, we arrive at

$$(-\omega^2 + \omega_{0,i}^2 - i\omega b_{0,i})p_i = \frac{q_i^2}{m_i}\left[G_0(\vec{r}_i-\vec{r}_s) + \sum_{j=1,j\neq i}^N \frac{\mu_0\omega^2}{4\pi}\right.$$

$$\left. \times iH_0^{(1)}(k|\vec{r}_i-\vec{r}_j|)p_i\right].$$
(7)

This set of linear equations can be solved numerically for $p_i$. After $p_i$ are obtained, the total field at any space point can be readily calculated from

$$E_z(\vec{r}) = G_0(\vec{r}-\vec{r}_s) + \sum_{j=1}^N G_j(\vec{r}-\vec{r}_j).$$
(8)

In the standard approach to wave localization, waves are said to be localized when the square modulus of the field $|E(\vec{r})|^2$, representing the wave energy, is spatially localized after the trivial cylindrically spreading effect is eliminated. Obviously, this is equivalent to say that the further away is the dipole from the source, the smaller its oscillation amplitude, expected to follow an exponentially decreasing pattern.

To this end, it is instructive to point out that an alternative two-dimensional dipole model was deviced previously by Rusek and Orlowski [14]. The authors derived a set of linear algebraic equations, which is similar in form to the above Eq. (7). However, there are some fundamental differences between the two models. In Ref. [14], the interaction between dipoles and the external field is derived by the energy conservation, while in the present case the coupling is determined without ambiguity by the Newton's second law. The former leads to an undetermined phase factor. According to, e.g., Refs. [24,25], the energy conservation can only give the radiation factor in Eq. (2). We would also like to point out that the set of couple equations in Eq. (7) is similar in spirit to the tight-binding model used to study the electronic localization [3,27].

There are several ways to introduce randomness to Eq. (7). For example, the disorder may be introduced by randomly varying such properties of individual dipoles as the charge, the mass or the two combined. This is the most common way that the disorder is introduced into the tight-binding model for electronic waves [27]. In the present study, the disorder is brought in by the random distribution of the dipoles.

Before moving to solve Eq. (7) for the phenomenon of localization of EM waves, we discuss a general property of wave localization. The energy flow of EM waves is $\vec{J}\sim\vec{E}\times\vec{H}$. By invoking the Maxwell equations to relate the electric and magnetic fields, we can derive that the time averaged energy flow is

$$\langle\vec{J}\rangle_t \equiv \frac{1}{T}\int_0^T dt\vec{J} \sim |\vec{E}|^2\nabla\theta,$$
(9)

where the electrical field is written as $\vec{E}=\vec{e}_E|\vec{E}|e^{i\theta}$, with $\vec{e}_E$ denoting the direction, $|\vec{E}|$ and $\theta$ being the amplitude and the phase, respectively. It is clear from Eq. (9) that when $\theta$ is constant, at least by spatial domains, while $|\vec{E}|\neq 0$, the flow

would come to a stop and the energy will be localized or stored in the space. In the localized state, a source can no longer radiate energies. Alternatively, we can write the oscillation of the dipoles as $p_i = |p_i| e^{i\theta_i}$. By studying the square modulus of $p_i$ in the form of $|\vec{r}_i - \vec{r}_s||p_i|^2$, and its phase $\theta_i$, we can also investigate the localization of EM waves. Note here that the factor $|\vec{r}_i - \vec{r}_s|$ is to eliminate the cylindrical spreading effect in 2D as can be seen from the expansion of the Hankel function $|H_0^{(1)}(x)|^2 \sim 1/x$. That the phase $\theta$ is constant implies that a coherence behavior appears in the system, i.e., the localized state is a phase-coherent state, as previously discussed [26]. It is a unique feature of wave localization, and has also been shown to be related to electronic localization (e.g., Ref. [28]).

For simplicity yet without losing generality, assume that all the dipoles are identical and they are randomly distributed within a square area. The source is located at the center (set to be the origin) of this area. For convenience, we make Eq. (7) nondimensional by scaling the frequency by the resonance frequency of a single dipole $\omega_0$. This will lead to two independent nondimensional parameters $b = (q^2 \mu_0/4m)$ and $b_0' = (\omega/\omega_0)(b_0/\omega_0)$. Both parameters may be adjusted in the experiment. For example, the factor $b_0$ can be modified by coating layered structures around the dipoles [24]. Then Eq. (7) becomes simply

$$(-f^2 + 1 - ib_0')p_i = ibf^2 \left[ p_0 H_0^{(1)}(k|\vec{r}_i - \vec{r}_s|) \right.$$

$$\left. + \sum_{j=1, j \neq i}^{N} p_j H_0^{(1)}(k|\vec{r}_i - \vec{r}_j|) \right], \quad (10)$$

with $f = (\omega/\omega_0)$. Equation (10) can be solved numerically for $p_i$ and then the total field is obtained through Eqs. (3), (5), and (8). In the calculation, we scale all lengths by the averaged distance between dipoles $d$. In this way, the frequency $\omega$ always enters as $kd$ and the natural frequency $\omega_0$ as $k_0 d$. We find that all the results shown below are only dependent on parameters $b$, $b_0/\omega_0$, and the ratio $\omega/\omega_0$ or equivalently $k/k_0 = (kd)/(k_0 d)$. Such a simple scaling property may facilitate designing experiments.

We have first computed the transmitted intensity averaged over the random distribution of the dipoles as a function of nondimensional frequency $kd$. The results indicate that the natural frequency of the dipoles is altered by the multiple scattering of EM waves, and is shifted towards the lower frequency end. We also find that the transmission is significantly suppressed in a range of frequencies slightly above the natural frequency of a single dipole, suggesting the strong localization in this range of frequencies. This helps us to search for the frequencies where the strong localization is most prominent. We show an example below.

By setting $b = b_0/\omega_0 = 10^{-3}$ and $k_0 d = 1$, we find a strong localization of EM waves at, for instance, $\omega/\omega_0 = 1.008$. In this case, the natural frequency of the dipoles is shifted to around $0.9\omega_0$. A typical picture of wave localization is shown in the bottom part of Fig. 1 for an arbitrary random distribution of 400 dipoles. To describe the phase behavior of
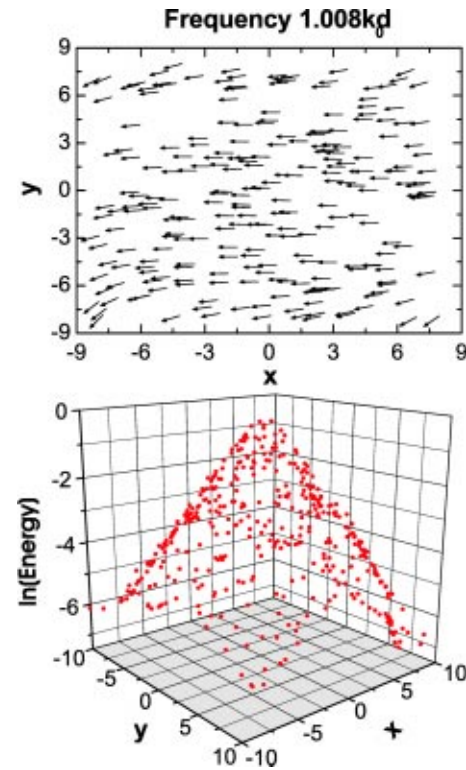


FIG. 1. Top: The phase diagram for the two-dimensional phase vectors defined in the text; the phase of the source is assumed to be zero. Each vector is located at the site of the dipole; thus the locations of the phase vectors also denote the random distribution of the dipoles. Bottom: The spatial distribution of energy ($\sim |E|^2$). Here the geometrical factor has been removed. Here the energy has been nondimensionalized and is plotted in the natural log scale.

the system, we assign a unit phase vector, $\vec{u}_i = \cos\theta_i \vec{e}_x + \sin\theta_i \vec{e}_y$ to the oscillation phase $\theta_i$ of the dipoles. Here $\vec{e}_x$ and $\vec{e}_y$ are unit vectors in the $x$ and $y$ directions, respectively. These phase vectors are represented by a phase diagram in the $x$-$y$ plane with the phase vector $\vec{u}_i$ being located at the dipole to which the phase $\theta_i$ is associated. The phase behavior of the system is depicted by the top portion of Fig. 1.

Here it is clearly shown that the field energy is strongly localized near the transmitting source, and, as expected, the energy decreases almost exponentially along the radial direction. Meanwhile, the system reveals an in-phase phenomenon: nearly all the phase vectors of the dipoles point to the same direction, exactly opposite to the phase vector of the source, i.e., the dipoles tend to oscillate out of phase with the source. It is easy to see that such a coherent behavior effectively prevents waves from propagation. The picture represented by Fig. 1 fully complies with the general description of wave localization stated above. The energy localization in Fig. 1 is also in qualitative agreement with that observed in the microwave localizaton [11,12] and the acoustic localization in water with air cylinders [26].

We note from Fig. 1 that near the sample boundary, the phase vectors start to point to different directions. This is because the numerical simulation is carried out for a finite sample size. For a finite system, the energy can leak out at
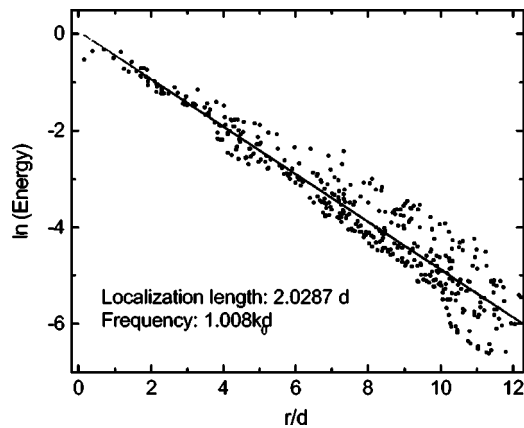
FIG. 2. Energy versus the distance away from the source. Again we removed the cylindrical spreading factor. Here the energy has been nondimensionalized and is plotted in the natural log scale.

the boundary, resulting in disappearance of the phase coherence [29]. When enlarging the sample size by adding more dipoles while keeping the averaged distance between dipoles fixed, we observe that the area showing the perfect phase coherence will increase accordingly. Another factor affecting the phase coherence behavior, and subsequently the wave localization, is the damping factor. When the absorption is added, the in-phase behavior will be degraded, and the waves will become delocalized gradually, in agreement with the

previous simulation on acoustic localization in bubbly water [29]. Though degrading, we find that the localization can sustain a substantial variation in the damping factor, making the system a good candidate for observing the localization phenomenon.

To find the localization length in Fig. 1, we plot the total energy in all directions as a function of the distance from the source. The results are presented by Fig. 2. Here, the numerical data are shown by the black squares, and the result fitted from the least squares method is shown by the solid line; the deviation from the line reflects the fluctuation due to the random distribution. It shows that after removing the spreading factor, the data can be fitted by $e^{-r/\xi}$. From the slope of the solid line, the localization length $\xi$ is estimated as around $2.02d$, which is in the vicinity of the localization length $1.6d$ estimated experimentally for microwave localization in 2D dielectric lattices [12].

Finally, we point out that as expected, further numerical results show that the above localization behavior holds true for any random configuration of the dipoles.

In summary, we have presented a model system to study the localization of EM waves in 2D random media. The results show that spatially localized EM waves can be realized in this simple but realistic disordered system.

[1] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978).

[2] N.M. Lawandy, R.M. Balachandran, A.S.L. Gomes, and E. Sauvain, Nature (London) **368**, 436 (1994).

[3] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).

[4] S. John, Phys. Rev. Lett. **53**, 2169 (1984).

[5] C.H. Hodges and J. Woodhouse, J. Acoust. Soc. Am. **74**, 894 (1983).

[6] T.R. Kirkpatrick, Phys. Rev. B **31**, 5746 (1985).

[7] S. He and J.D. Maynard, Phys. Rev. Lett. **57**, 3171 (1986).

[8] C.A. Condat, J. Acoust. Soc. Am. **83**, 441 (1988).

[9] D. Sornette and B. Souillard, Europhys. Lett. **7**, 269 (1988).

[10] A.Z. Genack and N. Garcia, Phys. Rev. Lett. **66**, 2064 (1991).

[11] R. Dalichaouch, J.P. Armstrong, S. Schultz, P.M. Platzman, and S.L. McCall, Nature (London) **354**, 53 (1991).

[12] S.L. McCall, P.M. Platzman, R. Dalichaouch, D. Smith, and S. Schultz, Phys. Rev. Lett. **67**, 2017 (1991).

[13] A. Lagendijk and B.A. van Tiggelen, Phys. Rep. **270**, 143 (1996).

[14] M. Rusek and A. Orlowski, Phys. Rev. E **51**, R2763 (1995).

[15] M.M. Sigalas, C.M. Soukoulis, C.-T. Chan, and D. Turner, Phys. Rev. B **53**, 8340 (1996).

[16] A.A. Asatryan *et al.*, Phys. Rev. B **57**, 13 535 (1998).

[17] P. Sheng, *Introduction to Wave Scattering, Localization and Mesoscopic Phenomena* (Academic Press, New York, 1995).

[18] Y. Kuga and A. Ishimaru, J. Opt. Soc. Am. A **1**, 831 (1984); M. van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985); P.E. Wolf and G. Maret, *ibid.* **55**, 2696 (1985); A. Tourin *et al.*, *ibid.* **79**, 3637 (1997); M. Torres, J.P. Adrados, and F.R. Montero de Espinosa, Nature (London) **398**, 114 (1999).

[19] D.S. Wiersma, P. Bartolini, A. Lagendijk, and R. Roghini, Nature (London) **390**, 671 (1997).

[20] A.A. Chabanov, M. Stoytchev, and A.Z. Genack, Nature (London) **404**, 850 (2000).

[21] F. Scheffold, R. Lenke, R. Tweer, and G. Maret, Nature (London) **398**, 206 (1999); D. Wiersma *et al.*, *ibid.* **398**, 207 (1999).

[22] H. Kuhn, J. Chem. Phys. **53**, 101 (1970).

[23] R. R. Chance, A. Prock, and R. Silbey, in *Advances in Chemical Physics*, edited by I. Prigogine and S. A. Rice (Wiley, New York, 1978), Vol. 37, pp. 1–65.

[24] T. Erdogan, K.G. Sullivan, and D.G. Hall, J. Opt. Soc. Am. B **10**, 391 (1993).

[25] C.-C. Wang and Z. Ye, Phys. Status Solidi A **174**, 527 (1999).

[26] E. Hoskinson and Z. Ye, Phys. Rev. Lett. **83**, 2734 (1999).

[27] R. Zallen, *The Physics of Amorphous Solids* (Wiley, New York, 1983).

[28] S.A. Gurtvitz, Phys. Rev. Lett. **85**, 812 (2000), and references therein.

[29] K.-X. Wang and Z. Ye, Phys. Rev. E **64**, 056607 (2001).